UNIVERSITY OF IBADAN

ODE

INTEGRATION BEE (2.0)

UI ODE-Integration Bee
Qualifying Stage (200 L)

Instructions for Participants

Thank you for choosing to participate in the UI ODE-Integration Bee Qualifying Stage. Please carefully read and follow these instructions:

1. Answer all questions.
2. Write your responses legibly and concisely. Use a clear and neat handwriting.
3. Use only the provided sheets for your answers. Ensure that your solutions are well-structured and organized.
4. Write your full name and matriculation number at the top of each page of your answer sheet.
5. Follow any specific instructions provided with individual questions.
6. Do not waste too much time on a question.
7. Be mindful of time. You will have 2 hours 30 minutes for the entire test.
8. If you have any questions or require clarification during the test, please raise your hand and wait for an invigilator to assist you.
9. Electronic devices, calculators, books, and any unauthorized aids are strictly prohibited during the test.
10. Maintain academic integrity. Do not discuss the content of the test with your fellow participants until the test is over.

This Qualifying stage aims to evaluate your understanding and problem-solving skills in the field of Ordinary Differential Equation (ODE). Good luck!.

Questions

Information for participants: The maximum points attainable for this test is 60 points. Take your time to read each questions carefully before you provide answers to them.

1. (8 points) Consider the series

$$\mathscr{H}(x) = \sum_{n=0}^{\infty} a_n x^{2n}, \text{ where } a_n = \frac{n!}{(2n)!}.$$

It often useful to find the recurrence relation involved, Here we have $a_n = \frac{a_{n-1}}{2(2n-1)}$. Observe that,

$$\mathscr{H}(x) = \sum_{n=0}^{\infty} x^{2n} a_n = 1 + \sum_{n=1}^{\infty} x^{2n} a_n$$

Taking the derivatives of both sides we obtain,

$$\mathscr{H}'(x) = \frac{1}{2}\sum_{n=1}^{\infty}(4n-2)a_n x^{2n-1} + \sum_{n=1}^{\infty} a_n x^{2n-1}$$

Thus, we have that

$$\mathscr{H}'(x) = \frac{1}{2}x\mathscr{H}(x) + \frac{1}{x}(\mathscr{H}(x) - 1)$$

Which is an excellent news, now show that

$$\mathscr{H}(x) = \frac{\sqrt{\pi}}{2}x \exp\left(\frac{x^2}{4}\right) erf\left(\frac{x}{2}\right) + 1$$

Remark. *We define*

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t.$$

2. (6 points) Consider an RL circuit with a resistor of 3Ω and an inductor of $0.01H$, powered by a voltage $E(t) = \sin(10t)$ V voltage source. Initially, the current through the resistor, $I(0) = 0$A. Calculate the following:

   a. The current $I(t)$ in the circuit as a function of time.
   b. The voltage across the inductor as a function of time.
   c. The voltage across the resistor as a function of time.

3. (6 points) Determine the solution of the first order differential equation,

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{t\omega + 2\omega - t - 2}{t\omega - 3\omega + t - 3}$$

4. (6 points) Consider the differential equation $ay'' + by' + cy = e^{kx}$, where a, b, c and k are constants. The auxiliary equation of the associated homogeneous equation is $am^2 + bm + c = 0$. If k is a root of the auxiliary equation of multiplicity two, show that we can find a particular solution of the form $y = Ax^2e^{kx}$, where $A = \frac{1}{2a}$.

5. (6 points) Solve the non-homogeneous differential equation

$$x^2y'' - 3xy' + 3y = 2x^5e^x$$

6. Solve the initial value problem $y'' + 4y = g(x)$, $y(0) = 1, y'(0) = 2$, where

$$g(x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{2}, \\ 0, & x > \frac{\pi}{2} \end{cases}$$

7. (6 points) Show that the solution of the initial value problem

$$y'' + y = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0$$

is

$$y = \int_{t_0}^{t} \sin(t-s)g(s)\,\mathrm{d}s$$

8. (4 points) Given that $z = f(x)$ and $y = g(x)$ satisfy the following differential equations

$$\frac{\mathrm{d}z}{\mathrm{d}x} + 2z = e^{-2x} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} + 2y = z,$$

a. Find z in the form $z = f(x)$.

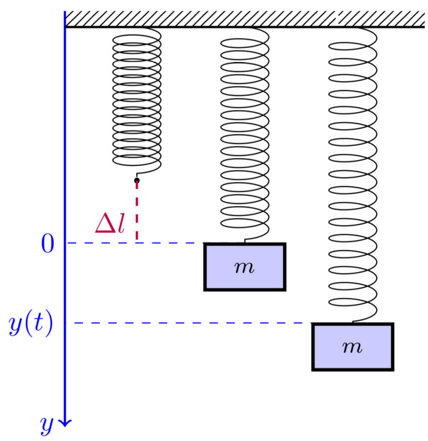
b. Express y in the form $y = g(x)$, given further that at $x = 0$, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

9. (4 points) Solve the differential equation,

$$\sin x\frac{\mathrm{d}y}{\mathrm{d}x} + (\cos x)y = \sin\left(x^2\right).$$

10. (8 points) A spring is an object that when deformed by an amount Δl creates a force $F_s = -k\Delta l$, with $k > 0$. Consider a spring-body system as shown below. A spring is fixed to a ceiling and hangs vertically with a natural length l. It stretches by Δl when a body with mass m is attached to its lower end, just as in the middle spring in the diagram below. We assume that the weight m is small enough so that the spring is not damaged. This means that the spring acts like a normal spring, whenever it is deformed by an amount Δl it makes a force proportional and opposite to the deformation, $F_{s0} = -k\Delta l$. Now, show that a spring-body system with spring constant k, body mass m, at rest with a spring deformation Δl, within the rage where the spring acts like a spring, satisfies

$$mg = k\Delta l.$$



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