University of Ibadan

ODE

INTEGRATION BEE

UI ODE-Integration Bee

Qualifying Stage (200 L)

Instructions for Participants

Thank you for choosing to participate in the UI ODE-Integration Bee Qualifying Stage. Please carefully read and follow these instructions:

1. Answer all questions.
2. Write your responses legibly and concisely. Use a clear and neat handwriting.
3. Use only the provided sheets for your answers. Ensure that your solutions are well-structured and organized.
4. Write your full name and matriculation number at the top of each page of your answer sheet.
5. Follow any specific instructions provided with individual questions.
6. Do not waste too much time on a question.
7. Be mindful of time. You will have 2 hours 30 minutes for the entire test.
8. If you have any questions or require clarification during the exam, please raise your hand and wait for an invigilator to assist you.
9. Electronic devices, calculators, books, and any unauthorized aids are strictly prohibited during the test.
10. Maintain academic integrity. Do not discuss the content of the test with your fellow participants until the test is over.

This Qualifying Stage aims to evaluate your understanding and problem-solving skills in the field of ordinary differential equation (ODE). Good luck!

Questions

Information for participants: The maximum points attainable for this test is 60 points. Take your time to read each question carefully before you provide answers to them.

1. (8 points) The dead body of a man was found by the police at 1:00 AM in a room that was maintained at 69°F. The body is 75°F when it was found, and had cooled to 73°F at 2:00 AM. Assist the police by estimating his time of death, assuming a living body maintains a temperature of 98.6°F.

2. (6 points) Solve the first order ordinary differential equation

$$y = xy' + \frac{1}{9y'}.$$

3. (6 points) Solve the first order ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y^2 - x^2}{5xy}.$$

4. (6 points) Solve the second order ordinary differential equation

$$y'' - 10y' + 25y = \mathrm{e}^{5t}.$$

5. (6 points) The objective of this problem is to convey to participants the idea that certain definite integrals can be evaluated through transformation into ordinary differential equations. To this end, consider a function of two variables, $f(a,b)$, defined as follows:

$$f(a,b) = \int_0^\infty \mathrm{e}^{-ax^2} \cos(bx)\,\mathrm{d}x,$$

where $\mathrm{Re}(a) > 0$ and $b \in \mathbb{C}$.

   (i) Show that

$$\frac{\partial f(a,b)}{\partial b} + \frac{b}{2a} f(a,b) = 0.$$

   (ii) Solve the differential equation in (i) and use the fact that

$$f(a,0) = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

   to establish that

$$f(a,b) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\mathrm{e}^{-\frac{b^2}{4a}}.$$

6. Form a homogeneous differential equation associated with the function

$$y(t) = a\mathrm{e}^t + bt\mathrm{e}^t + c\mathrm{e}^{2t},$$

where a, b and c are arbitrary constants.

7. (6 points) Determine the function $w(x, y)$, given that

$$\mathrm{d}w = \mathrm{e}^y \,\mathrm{d}x + (2y + x\mathrm{e}^y)\,\mathrm{d}y.$$

8. (6 points) Establish that the integrating factor of the first order ordinary differential equation

$$a\frac{\mathrm{d}y}{\mathrm{d}x} + by = f(x)$$

is given by $\mathrm{e}^{\frac{b}{a}x}$.

9. (6 points) In an electric circuit with resistance R (measured in ohms) and inductance L (measured in henrys), the dependence of the voltage $E(t)$ (measured in volts) and the current $I(t)$ (in amperes) is given by

$$E(t) = RI(t) + L\frac{\mathrm{d}I(t)}{\mathrm{d}t}.$$

Consider the situation when the voltage $E(t) = 9$ volts, and $I(0) = I_0$.

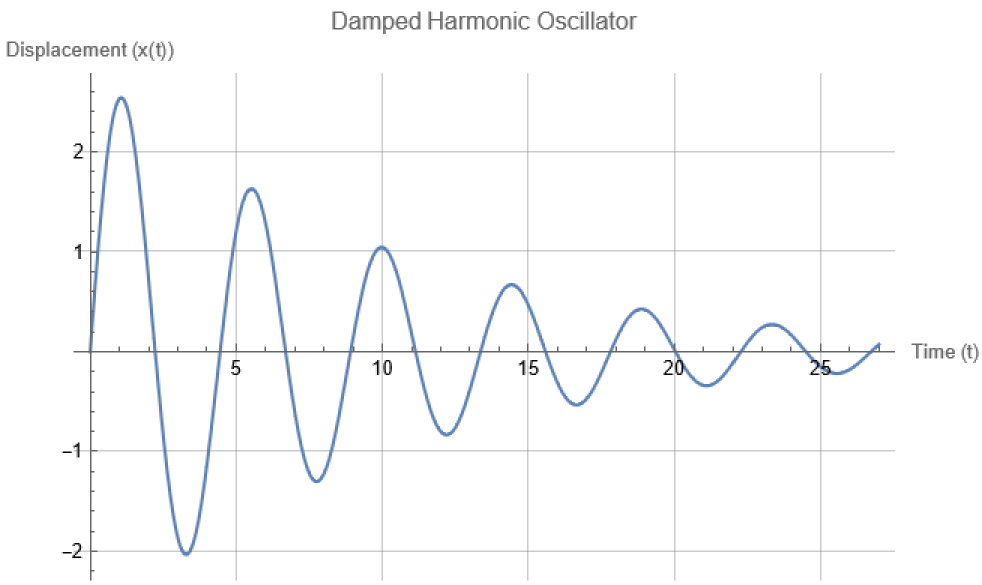
(i) Find an explicit representation for the current, $I(t)$.

(ii) Describe the behaviour of $I(t)$ as t rapidly increases.

10. (4 points) A damped harmonic oscillation modelled by the differential equation

$$5x''(t) + x'(t) + 10x(t) = 0, \quad x(0) = 0, x'(0) = 4,$$

is illustrated by the following diagram.



Determine $x(t)$.