University of Ibadan

ODE

INTEGRATION BEE

# UI ODE-Integration Bee

## Qualifying Stage (200 L)

## Instructions for Participants

Thank you for choosing to participate in the UI ODE-Integration Bee Qualifying Stage. Please carefully read and follow these instructions:

1. Answer all questions.
2. Write your responses legibly and concisely. Use a clear and neat handwriting.
3. Use only the provided sheets for your answers. Ensure that your solutions are well-structured and organized.
4. Write your full name and matriculation number at the top of each page of your answer sheet.
5. Follow any specific instructions provided with individual questions.
6. Do not waste too much time on a question.
7. Be mindful of time. You will have 2 hours 30 minutes for the entire test.
8. If you have any questions or require clarification during the exam, please raise your hand and wait for an invigilator to assist you.
9. Electronic devices, calculators, books, and any unauthorized aids are strictly prohibited during the test.
10. Maintain academic integrity. Do not discuss the content of the test with your fellow participants until the test is over.

This Qualifying Stage aims to evaluate your understanding and problem-solving skills in the field of ordinary differential equation (ODE). Good luck!

# Questions & Solutions

Information for participants: The maximum points attainable for this test is 60 points. Take your time to read each question carefully before you provide answers to them.

1. (8 points) The dead body of a man was found by the police at 1:00 AM in a room that was maintained at 69°F. The body was 75°F when it was found, and had cooled to 73°F at 2:00 AM. Assist the police by estimating his time of death, assuming a living body maintains a temperature of 98.6°F.

   **Solution.** By Newton's law of cooling, we have

$$\frac{\mathrm{d}T(t)}{\mathrm{d}t} = -k(T - T_r),$$

   where $T_r$ is the room temperature, and $k$ is a constant of proportionality. Separating variables, we have

$$\frac{\mathrm{d}T(t)}{T(t) - T_r} = -k\mathrm{d}t.$$

   Changing the variable $t$ to $\alpha$, we have

$$\frac{\mathrm{d}T(\alpha)}{T(\alpha) - T_r} = -k\mathrm{d}\alpha.$$

   Integrating both sides from $\alpha = 0$ to $t$, we have

$$\ln(T(\alpha) - T_r)\Big|_0^t = -kt,$$

$$\ln\left(\frac{T(t) - T_r}{T(0) - T_r}\right) = -kt.$$

   This implies

$$\frac{T(t) - T_r}{T(0) - T_r} = \mathrm{e}^{-kt},$$

$$T(t) - T_r = (T(0) - T_r)\,\mathrm{e}^{-kt},$$

$$T(t) = T_r + (T(0) - T_r)\,\mathrm{e}^{-kt}.$$

   From the information given, $T_r = 69$°F, $T(0) = 75$°F. Thus, we have

$$T(t) = 69 + (75 - 69)\,\mathrm{e}^{-kt}.$$

$$T(t) = 69 + 6\,\mathrm{e}^{-kt}.$$

   Since $T(1) = 73$, we have

$$73 = 69 + 6\,\mathrm{e}^{-k},$$

   and so

$$\frac{2}{3} = \mathrm{e}^{-k}.$$

   Therefore,

$$T(t) = 69 + 6\left(\frac{2}{3}\right)^t.$$

Given that $T(t) = 98.6°\text{F}$, we have

$$\frac{98.6 - 69}{6} = \left(\frac{2}{3}\right)^t.$$

This implies

$$t = \frac{\ln\left(\frac{98.6-69}{6}\right)}{\ln 2 - \ln 3} = -3.94.$$

Converting to minutes, we have

$$t = -3 + 0.94 \cdot 60 = -3{:}56{:}4.$$

The time of death of the man is

$$1{:}00\text{ AM} - 3{:}56{:}4 = 9{:}03{:}56\text{ PM}.$$

Hence, the time of death of the man is 9:03:56 PM (that is, 9:00 PM + 3 minutes + 56 seconds), 9:04 PM approximately.

2. (6 points) Solve the first order ordinary differential equation

$$y = xy' + \frac{1}{9y'}.$$

**Solution.** Let $t(x) = y'$. Then

$$y = xt + \frac{1}{9t}.$$

Differentiating both sides with respect to $x$, we have

$$y' = t + xt' - \frac{1}{9t^2}t'. \tag{1}$$

Substituting $y' = t$ on the left-hand side of (1), we have

$$t = t + xt' - \frac{1}{9t^2}t'$$

Simplifying, we have

$$xt' - \frac{1}{9t^2}t' = 0, \quad \left(x - \frac{1}{9t^2}\right)t' = 0.$$

This implies $t' = 0$, that is $t = c_1$, or

$$x = \frac{1}{9t^2}, \quad t = \frac{1}{3\sqrt{x}}.$$

Substituting into $y' = t$, we have $y' = c_1$ or $y' = \frac{1}{3\sqrt{x}}$. From the first resulting equation, we have

$$y = c_1x + c_2,$$

where $c_1$ and $c_2$ are arbitrary constants. Differentiating, we have $y' = c_1$. Substituting $y$ and $y'$ into the given differential equation, we have

$$c_1x + c_2 = c_1x + \frac{1}{9c_1},$$

$$c_1^2 x + c_1 c_2 = c_1^2 x + \frac{1}{9}.$$

We find that

$$c_2 = \frac{1}{9c_1}.$$

Therefore,

$$y = c_1 x + \frac{1}{9c_1}$$

From the second resulting equation, we have

$$y = \frac{2}{3}\sqrt{x} + c_2$$

Differentiating, we have $y' = \frac{1}{3\sqrt{x}}$. Substituting $y$ and $y'$ into the given ODE, we have

$$\frac{2}{3}\sqrt{x} + c_2 = x \cdot \frac{1}{3\sqrt{x}} + \frac{1}{9} \cdot 3\sqrt{x} = \frac{2}{3}\sqrt{x}.$$

This implies

$$c_2 = 0.$$

Hence

$$y = Cx + \frac{1}{9C},$$

where $C$ is a nonzero arbitrary constant, or

$$y = \frac{2}{3}\sqrt{x}$$

are solutions of the first order equation.

3. (6 points) Solve the first order ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5y^2 - x^2}{5xy}.$$

**Solution.** Substituting $y = vx$, we have

$$y' = v + xv',$$

$$v + xv' = \frac{5v^2 - 1}{5v},$$

$$xv' = \frac{-1}{5v}.$$

Separating variables, we have

$$5v\,\mathrm{d}v = -\frac{\mathrm{d}x}{x}$$

$$\frac{5v^2}{2} = -\ln x + c_1,$$

where $c_1$ is an arbitrary constant. Simplifying, we have

$$v^2 = -\frac{2}{5}\ln x + C$$

$$v = \sqrt{C - \frac{2}{5}\ln x}.$$

Hence,

$$y = x\sqrt{\frac{5C - 2}{5}\ln x},$$

where $C$ is an arbitrary constant.

4. (6 points) Solve the second order ordinary differential equation

$$y'' - 10y' + 25y = \mathrm{e}^{5t}.$$

**Solution.** The auxiliary equation is $m^2 - 10m + 25 = 0$. This implies $m = 5$ twice, and so complementary factor is $y = c_1 \mathrm{e}^{5t} + c_2 t\,\mathrm{e}^{5t}$. Let $y = Ct^2\,\mathrm{e}^{5t}$ be the particular solution of the differential equation. Then

$$\begin{aligned} y' &= 2Ct\,\mathrm{e}^{5t} + 5Ct^2\,\mathrm{e}^{5t} \\ y'' &= 2C\,\mathrm{e}^{5t} + 10Ct\,\mathrm{e}^{5t} + 10Ct\,\mathrm{e}^{5t}\,\mathrm{e}^{5t} + 25Ct^2\,\mathrm{e}^{5t} \\ &= 2C\,\mathrm{e}^{5t} + 20Ct\,\mathrm{e}^{5t} + 25Ct^2\,\mathrm{e}^{5t}. \end{aligned}$$

Substituting these into the differential equation, we have

$$\begin{aligned} y'' - 10y' + 25y &= 2C\,\mathrm{e}^{5t} + 20Ct\,\mathrm{e}^{5t} + 25Ct^2\,\mathrm{e}^{5t} - 20Ct\,\mathrm{e}^{5t} - 50Ct^2\,\mathrm{e}^{5t} + 25Ct^2\,\mathrm{e}^{5t} \\ &= 2C\,\mathrm{e}^{5t} = \mathrm{e}^{5t}. \end{aligned}$$

Thus, we have $2C\,\mathrm{e}^{5t} = \mathrm{e}^{5t}$, and so, $C = \frac{1}{2}$. Therefore, the particular solution is

$$y = \frac{1}{2}t^2\,\mathrm{e}^{5t}.$$

Hence, the general solution is

$$y = c_1\,\mathrm{e}^{5t} + c_2 t\,\mathrm{e}^{5t} + \frac{1}{2}t^2\,\mathrm{e}^{5t},$$

where $c_1$ and $c_2$ are arbitrary constants.

5. (6 points) The objective of this problem is to convey to participants the idea that certain definite integrals can be evaluated through transformation into ordinary differential equations. To this end, consider a function of two variables, $f(a,b)$, defined as follows:

$$f(a,b) = \int_0^\infty \mathrm{e}^{-ax^2}\cos(bx)\,\mathrm{d}x,$$

where $\mathrm{Re}(a) > 0$ and $b \in \mathbb{C}$.

(i) Show that

$$\frac{\partial f(a,b)}{\partial b} + \frac{b}{2a}f(a,b) = 0.$$

(ii) Solve the differential equation in (i) and use the fact that

$$f(a,0) = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

to establish that

$$f(a,b) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\mathrm{e}^{-\frac{b^2}{4a}}.$$

**Solution.** (i) Evaluating the partial derivative, then integrating by parts, we have

$$\begin{aligned}\frac{\partial f(a,b)}{\partial b} &= -\int_0^\infty x\mathrm{e}^{-ax^2}\sin(bx)\,\mathrm{d}x \\ &= \int_0^\infty \mathrm{d}\left(\frac{\mathrm{e}^{-ax^2}}{2a}\right)\sin(bx) \\ &= \left.\frac{\mathrm{e}^{-ax^2}\sin(bx)}{2a}\right|_0^\infty - \frac{b}{2a}\int_0^\infty \mathrm{e}^{-ax^2}\cos(bx)\,\mathrm{d}x \\ &= -\frac{b}{2a}\int_0^\infty \mathrm{e}^{-ax^2}\cos(bx)\,\mathrm{d}x \\ &= -\frac{b}{2a}f(a,b).\end{aligned}$$

Hence,

$$\frac{\partial f(a,b)}{\partial b} + \frac{b}{2a}f(a,b) = 0.$$

(ii) **Method 1:** The integrating factor is $\mathrm{e}^{\int \frac{b}{2a}\,\mathrm{d}b} = \mathrm{e}^{\frac{b^2}{4a}}$. Therefore,

$$\frac{\partial}{\partial b}\left(f(a,b)\mathrm{e}^{\frac{b^2}{4a}}\right) = 0.$$

Integrating both sides with respect to $b$, we have

$$f(a,b)\mathrm{e}^{\frac{b^2}{4a}} = C,$$

which implies

$$f(a,b) = C\mathrm{e}^{-\frac{b^2}{4a}}.$$

**Method 2:** Separating variables, we have

$$\begin{aligned}\frac{\partial f(a,b)}{\partial b} + \frac{b}{2a}f(a,b) &= 0, \\ \frac{\partial f(a,b)}{\partial b} &= -\frac{b}{2a}f(a,b), \\ \frac{\partial f(a,b)}{f(a,b)} &= -\frac{b}{2a}\partial b, \\ \frac{\mathrm{d}f(a,b)}{f(a,b)} &= -\frac{b}{2a}\,\mathrm{d}b.\end{aligned}$$

Integrating both sides, we have

$$\ln f(a,b) = -\frac{b^2}{4a} + c_1,$$

where $c_1$ is an arbitrary constant. This implies

$$f(a,b) = C\mathrm{e}^{-\frac{b^2}{4a}},$$

where $C$ is an arbitrary constant. Substituting $b = 0$, we have

$$f(a,0) = C = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

Hence,

$$f(a,b) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\mathrm{e}^{-\frac{b^2}{4a}}.$$

6. (6 points) Form a homogeneous differential equation associated with the function

$$y(t) = a\mathrm{e}^t + bt\mathrm{e}^t + c\mathrm{e}^{2t},$$

where $a$, $b$ and $c$ are arbitrary constants.

**Solution.** The auxiliary equation associated with the homogeneous differential equation is given by

$$(m-1)^2(m-2) = 0.$$

Expanding this, we have

$$(m^2 - 2m + 1)(m-2) = m^3 - 4m^2 + 5m - 2 = 0.$$

Hence, the homogeneous differential equation is

$$y''' - 4y'' + 5y' - 2y = 0.$$

7. (6 points) Determine the function $w(x, y)$, given that

$$\mathrm{d}w = \mathrm{e}^y\,\mathrm{d}x + (2y + x\mathrm{e}^y)\,\mathrm{d}y.$$

**Solution.** The differential equation is exact as

$$\frac{\partial}{\partial y}(\mathrm{e}^y) = \mathrm{e}^y,$$

and

$$\frac{\partial}{\partial x}(2y + x\mathrm{e}^y) = \mathrm{e}^y.$$

Therefore,

$$\begin{aligned} \frac{\partial w}{\partial x} &= \mathrm{e}^y \\ w &= \int \mathrm{e}^y\,\mathrm{d}x \\ w &= x\mathrm{e}^y + f(y) \\ \frac{\partial w}{\partial y} &= x\mathrm{e}^y + f'(y) \end{aligned}$$

Since

$$\frac{\partial w}{\partial y} = 2y + x\mathrm{e}^y,$$

we have

$$f(y) = y^2 + C.$$

Hence,

$$w(x, y) = y^2 + x\mathrm{e}^y + C,$$

where $C$ is an arbitrary constant.

8. (6 points) Establish that the integrating factor of the first order ordinary differential equation

$$a\frac{\mathrm{d}y}{\mathrm{d}x} + by = f(x)$$

is given by $\mathrm{e}^{\frac{b}{a}x}$.

**Solution.** Dividing through by $a$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{b}{a}y = \frac{f(x)}{a}.$$

We seek a function $g(x)$ such that

$$g(x)\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)\frac{b}{a}y = g(x)\frac{f(x)}{a}, \tag{2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)y\right) = \frac{f(x)}{a}. \tag{3}$$

Differentiating, we have

$$g(x)\frac{\mathrm{d}y}{\mathrm{d}x} + g'(x)y = g(x)\frac{f(x)}{a}.$$

Comparing (2) and (3), we have

$$g'(x) = g(x)\frac{b}{a}.$$

Separating variables and integrating, we have

$$\begin{aligned}
\frac{g'(x)}{g(x)} &= \frac{b}{a}, \\
\int \frac{g'(x)}{g(x)}\mathrm{d}x &= \int \frac{b}{a}\mathrm{d}x, \\
\ln g(x) &= \frac{b}{a}x + c_1 \\
g(x) &= C\,\mathrm{e}^{\frac{b}{a}x},
\end{aligned}$$

where $c_1$ and $C$ are arbitrary constants. Multiplying through by this expression for $g(x)$, we have

$$C\,\mathrm{e}^{\frac{b}{a}x}\frac{\mathrm{d}y}{\mathrm{d}x} + C\,\mathrm{e}^{\frac{b}{a}x}\frac{b}{a}y = \frac{C}{a}f(x)\,\mathrm{e}^{\frac{b}{a}x},$$

and so, $C$ cancels out. Hence, the integrating factor is $\mathrm{e}^{\frac{b}{a}x}$.

9. (6 points) In an electric circuit with resistance $R$ (measured in ohms) and inductance $L$ (measured in henrys), the dependence of the voltage $E(t)$ (measured in volts) and the current $I(t)$ (in amperes) is given by

$$E(t) = RI(t) + L\frac{\mathrm{d}I(t)}{\mathrm{d}t}.$$

Consider the situation when the voltage $E(t) = 9$ volts, and $I(0) = I_0$.

(i) Find an explicit representation for the current, $I(t)$.

(ii) Describe the behaviour of $I(t)$ as $t$ rapidly increases.

**Solution.** (i) Since $E(t) = 9$ volts, we have

$$9 = RI(t) + L\frac{\mathrm{d}I(t)}{\mathrm{d}t}.$$

Rearranging, we have

$$\begin{aligned}\frac{9}{L} &= \frac{R}{L} + \frac{\mathrm{d}I(t)}{\mathrm{d}t},\\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} + \frac{R}{L} &= \frac{9}{L}.\end{aligned}$$

The integrating factor is $\mathrm{e}^{\frac{R}{L}t}$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(I\,\mathrm{e}^{\frac{R}{L}t}\right) = \frac{9}{L}\,\mathrm{e}^{\frac{R}{L}t}$$

Changing the variable $t$ to $\alpha$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(I\,\mathrm{e}^{\frac{R}{L}\alpha}\right) = \frac{9}{L}\,\mathrm{e}^{\frac{R}{L}\alpha}$$

Integrating both sides from $\alpha = 0$ to $t$, we have

$$\begin{aligned}I(t)\,\mathrm{e}^{\frac{R}{L}t} - I(0) &= \frac{9}{L}\cdot\frac{L}{R}\left(\mathrm{e}^{\frac{R}{L}t} - 1\right)\\ I(t)\,\mathrm{e}^{\frac{R}{L}t} &= I_0 + \frac{9}{R}\left(\mathrm{e}^{\frac{R}{L}t} - 1\right)\\ I(t) &= I_0\,\mathrm{e}^{-\frac{R}{L}t} + \frac{9}{R}\left(1 - \mathrm{e}^{-\frac{R}{L}t}\right).\end{aligned}$$

Hence,

$$I(t) = \left(I_0 - \frac{9}{R}\right)\mathrm{e}^{-\frac{R}{L}t} + \frac{9}{R}.$$

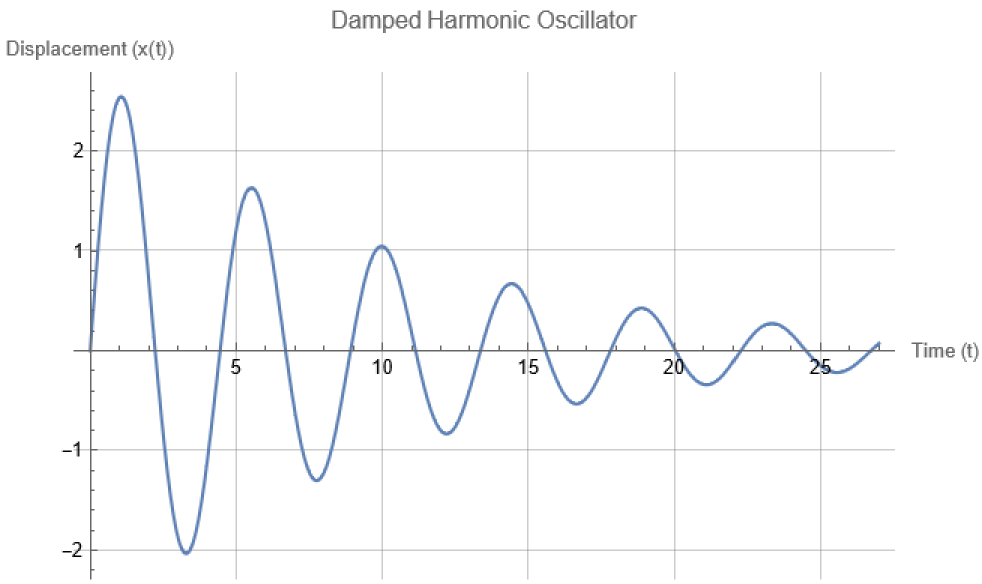
(ii) As $t \to \infty$, the current becomes

$$I(t) = \frac{9}{R}\ \text{ampere(s)}.$$

10. (4 points) A damped harmonic oscillation modelled by the differential equation

$$5x''(t) + x'(t) + 10x(t) = 0, \quad x(0) = 0, x'(0) = 4,$$

is illustrated by the following diagram.



Determine $x(t)$.

**Solution.** The auxiliary equation is given by

$$5m^2 + m + 10 = 0.$$

By the quadratic formula

$$m = \frac{-1 \pm \sqrt{1-200}}{10} = \frac{-1 \pm \mathrm{i}\sqrt{199}}{10},$$

where $\mathrm{i} = \sqrt{-1}$. The general solution of the differential equation is given by

$$x(t) = \mathrm{e}^{-\frac{t}{10}} \left( c_1 \sin\left(\frac{\sqrt{199}}{10}t\right) + c_2 \cos\left(\frac{\sqrt{199}}{10}t\right)\right),$$

where $c_1$ and $c_2$ are arbitrary constants. Since $x(0) = 0$, we have

$$c_2 = 0.$$

Therefore,

$$x(t) = c_1\,\mathrm{e}^{-\frac{t}{10}} \sin\left(\frac{\sqrt{199}}{10}t\right).$$

Differentiating, we have

$$x'(t) = -\frac{c_1}{10}\,\mathrm{e}^{-\frac{t}{10}} \sin\left(\frac{\sqrt{199}}{10}t\right) + \frac{c_1\sqrt{199}}{10} \cos\left(\frac{\sqrt{199}}{10}t\right).$$

Since $x'(0) = 4$, we have

$$4 = \frac{c_1\sqrt{199}}{10},$$

and so,

$$c_1 = \frac{40}{\sqrt{199}}.$$

Hence, the solution of the initial value problem is given by

$$x(t) = \frac{40}{\sqrt{199}}\,\mathrm{e}^{-\frac{t}{10}} \sin\left(\frac{\sqrt{199}}{10}t\right).$$