University of Ibadan

ODE

INTEGRATION BEE

UI ODE-Integration Bee

Qualifying Stage (100 L)

Instructions for Participants

Thank you for choosing to participate in the UI ODE-Integration Bee Qualifying Stage. Please carefully read and follow these instructions:

1. Answer all questions.
2. Write your responses legibly and concisely. Use a clear and neat handwriting.
3. Use only the provided sheets for your answers. Ensure that your solutions are well-structured and organized.
4. Write your full name and matriculation number at the top of each page of your answer sheet.
5. Follow any specific instructions provided with individual questions.
6. Do not waste too much time on a question.
7. Be mindful of time. You will have 2 hours 30 minutes for the entire test.
8. If you have any questions or require clarification during the test, please raise your hand and wait for an invigilator to assist you.
9. Electronic devices, calculators, books, and any unauthorized aids are strictly prohibited during the test.
10. Maintain academic integrity. Do not discuss the content of the test with your fellow participants until the test is over.

This Qualifying Stage aims to evaluate your understanding and problem-solving skills in the field of integration. Good luck!

Questions

Information for participants: The maximum points attainable for this test is 60 points. Take your time to read each question carefully before you provide answers to them.

1. (8 points) The Basel problem, posed by Pietro Mengoli in 1650 and famously solved by Leonhard Euler in 1734, sought to find the exact value of

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

This enigmatic puzzle baffled mathematicians for decades, inviting minds to unravel the mysterious convergence of the infinite series. Euler's approach involves the comparison of the x^2 term in the infinite product and infinite series of the $\sin x$. This question aims to explore a different proof.

Using the substitutions in the following order

$$x = \sin t, \quad u = \tan \frac{t}{2}, \quad u + y = t, \quad \arccos \phi = \arctan \left(\sqrt{\frac{1-\phi}{1+\phi}} \right),$$

show that

$$\int_{-1}^{1} \left(\int_{0}^{1} \frac{1}{(1 + yx)\sqrt{1-x^2}} \mathrm{d}x \right) \mathrm{d}y = \frac{\pi^2}{2}.$$

Note: By reversing the order of integration and utilising the substitution

$$v = \frac{1-x}{1+x},$$

one can conclude that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

2. (6 points) Evaluate the integral

$$\int \frac{x+1}{\sqrt{4x^2-1}} \,\mathrm{d}x.$$

3. (6 points) If $f(x)$ is an even function, that is, $f(x) = f(-x)$, with period a, that is, $f(x+a) = f(x)$, show that

$$\int_0^a f(x) \cos\left(\frac{\pi x}{a}\right) \mathrm{d}x = 0.$$

4. (6 points) Evaluate the integral

$$\lim_{n \to \infty} \int_0^n \frac{\sqrt{3n}}{5 + 7nt^2} \,\mathrm{d}t.$$

5. (6 points) Evaluate the integral

$$\int \frac{1}{\cos t - \sin t}\,\mathrm{d}t.$$

6. (6 points) Let $a > 0$. Evaluate the integral

$$\int_{-a}^{a} \frac{1}{(e^x + 1)\cos x}\,\mathrm{d}x.$$

7. (6 points) Evaluate the integral

$$\int_0^1 \frac{\arctan x}{1 + x^2}\,\mathrm{d}x.$$

8. (6 points) Evaluate the integral

$$\int \sqrt{\frac{x+1}{x+2}}\,\mathrm{d}x.$$

9. (6 points) Evaluate the integral

$$\int \frac{1}{\sqrt{9-x}\sqrt{9+x}}\,\mathrm{d}x.$$

10. (4 points) For what values of a is the integral

$$\int_0^{\sqrt[3]{a}} \frac{\mathrm{d}x}{x^4 + a}$$

defined?

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