UI ODE-Integration Bee

Qualifying Stage (100 L)

Instructions for Participants

Thank you for choosing to participate in the UI ODE-Integration Bee Qualifying Stage. Please carefully read and follow these instructions:

1. Answer all questions.
2. Write your responses legibly and concisely. Use a clear and neat handwriting.
3. Use only the provided sheets for your answers. Ensure that your solutions are well-structured and organized.
4. Write your full name and matriculation number at the top of each page of your answer sheet.
5. Follow any specific instructions provided with individual questions.
6. Do not waste too much time on a question.
7. Be mindful of time. You will have 2 hours 30 minutes for the entire test.
8. If you have any questions or require clarification during the test, please raise your hand and wait for an invigilator to assist you.
9. Electronic devices, calculators, books, and any unauthorized aids are strictly prohibited during the test.
10. Maintain academic integrity. Do not discuss the content of the test with your fellow participants until the test is over.

This Qualifying stage aims to evaluate your understanding and problem-solving skills in the field of integration. Good luck!.

Questions

Information for participants: The maximum points attainable for this test is 60 points. Take your time to read each questions carefully before you provide answers to them.

1. (8 points) The Riemann zeta function, $\zeta(s)$, defined for $\Re(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

is an extremely important special function in mathematics an physics, arising in definite integration and intimately connected with very deep results surrounding the prime number theorem. Notably, the domain of $\zeta(s)$ can be extended to the entire complex plane $\mathbb{C}$, with the exception of $s = 1$, via the Hermite integral representation, which is a direct application of the Abel-Plana formula. For $\Re(s) > 1$, an integral representation relating $\zeta(s)$ to the gamma function is

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, \mathrm{d}x,$$

where $\Gamma(s)$ denotes the gamma function, defined as $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t}\, \mathrm{d}t$. Some values of $\zeta(s)$ for even s are

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}.$$

A generalization of $\zeta(s)$ for even s is given in terms of Bernoulli numbers.
In Abdulsalam's article titled "*New Closed Forms for a Dilogarithmic Integral, Related Integrals, and Series,*" he proved that

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} \int_0^{\infty} \frac{x}{(x^2 + k^2)^2 (e^{2\pi x} - 1)}\, \mathrm{d}x \right) = \frac{\pi^4}{288} - \frac{\zeta(3)}{4}.$$

Justify the convergence of this series.

2. (6 points) Define a sequence $\{\zeta_n\}_{n=0}^{\infty}$ by $\zeta_0 = 1$, $\zeta_1 = 8$ and $\zeta_n = 2\zeta_{n-1} + \zeta_{n-2}$ for $n \geq 2$. The infinite sum

$$\sum_{n=1}^{\infty} \int_0^{\frac{2021\pi}{14}} \sin(\zeta_{n-1} x) \sin(\zeta_n x)\, \mathrm{d}x$$

can be expressed as an irreducible fraction $\frac{p}{q}$. Compute $p + q$.

3. (6 points) Evaluate

$$\int_0^1 \frac{x}{x\sqrt{x} + 1}\, \mathrm{d}x$$

4. (6 points) Compute the value of $20a + b$, where a and b are positive integers satisfying

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, \mathrm{d}x = \frac{\pi^b}{a}$$

5. (6 points) Given

$$\int_0^8 \frac{1}{\sqrt{1+\sqrt{1+x}}}\,\mathrm{d}x = \frac{a+b\sqrt{2}}{c}$$

such that $\gcd(a,b,c) = 1$. Find the value of $a+b+c$. (Note: $\gcd(a,b)$ means the greatest common divisor of a and b).

6. (6 points) Compute

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\pi e}}\,\mathrm{d}x$$

Hint: Use the substitution $x = \frac{\pi}{2} - y$.

7. (6 points) If f is continuously differentiable on $[0,a]$ for $a > 0$, and $f(a) = f(0) = b$, prove that

$$\int_0^a (f'(x))^2\,\mathrm{d}x \geq 2\int_0^a f(x)\,\mathrm{d}x - \left(2ab + \frac{a^3}{12}\right)$$

Hint: Consider the inequality $0 \leq \displaystyle\int_0^a \left(f'(x) + x - \frac{a}{2}\right)^2 \mathrm{d}x$.

8. (4 points) Determine whether the integral converges or diverges;

$$\int_{-\infty}^{\infty} \frac{1}{4+x^2}\,\mathrm{d}x$$

9. (4 points) Compute

$$\int_1^{\infty} \arctan\left(\frac{1}{x}\right)\mathrm{d}x$$

10. (8 points) Find the volume of the object generated when the area between $y = x^2$ and $y = x$ is rotated around the x-axis. This solid has a hole in the middle; we can compute the volume by subtracting the volume of the hole from the volume enclosed by the outer surface of the solid.
In the figure below we show the region that is rotated, the resulting solid with the front half cut away, the cone that forms the outer surface, the horn-shaped hole, and a cross-section perpendicular to the x-axis.