University of Ibadan

ODE

INTEGRATION BEE

UI ODE-Integration Bee

Final Stage

Instructions for Participants

Congratulations on qualifying for the UI ODE-Integration Bee Final Stage! Please carefully read and follow these instructions:

1. Answer all questions.
2. Write your responses legibly and concisely. Use a clear and neat handwriting.
3. Use only the provided sheets for your answers. Ensure that your solutions are well-structured and organized.
4. Write your full name and matriculation number at the top of each page of your answer sheet.
5. Follow any specific instructions provided with individual questions.
6. Do not waste too much time on a question.
7. Be mindful of time. You will have 2 hours 30 minutes for the entire test.
8. If you have any questions or require clarification during the test, please raise your hand and wait for an invigilator to assist you.
9. Electronic devices, calculators, books, and any unauthorized aids are strictly prohibited during the test.
10. Maintain academic integrity. Do not discuss the content of the test with your fellow participants until the test is over.

This Final Stage aims to assess your understanding and problem-solving skills in integration and its application. Good luck!

Questions

Information for participants: The maximum points attainable for this test is 60 points. Take your time to read each question carefully before you provide answers to them.

1. (6 points) The Hurwitz zeta function, a generalization of the Riemann zeta function, is defined as:

$$\zeta(s, z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s},$$

where $\Re s > 1$. The domain $\Re s > 1$ can be extended to $s \in \mathbb{C} \setminus \{1\}$ through analytic continuation, using for instance, the Hermite integral representation for $\zeta(s, z)$. In the Abel-Plana formula, given by

$$\sum_{k=0}^{\infty} \phi(k+m) = \frac{1}{2}\phi(m) + \int_m^{\infty} \phi(x)\,\mathrm{d}x + \mathrm{i}\int_0^{\infty} \frac{\phi(m+\mathrm{i}y) - \phi(m-\mathrm{i}y)}{\mathrm{e}^{2\pi y} - 1}\,\mathrm{d}y,$$

if we let $m = 0$, and suppose that as $n \to \infty$, $\phi(n) \to 0$, $\phi(n \pm \mathrm{i}y) \to 0$, we obtain

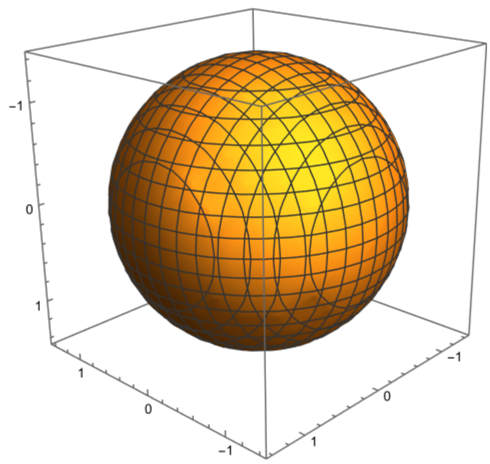
$$\sum_{k=0}^{\infty} \phi(k) = \frac{1}{2}\phi(0) + \int_0^{\infty} \phi(x)\,\mathrm{d}x + \mathrm{i}\int_0^{\infty} \frac{\phi(\mathrm{i}y) - \phi(-\mathrm{i}y)}{\mathrm{e}^{2\pi y} - 1}\,\mathrm{d}y.$$

Deduce Hermite's integral representation for $\zeta(s, z)$, given by

$$\zeta(s, z) = \frac{z^{-s}}{2} + \frac{z^{1-s}}{s-1} + 2\int_0^{\infty} \frac{\sin\left(s \arctan\left(x/z\right)\right)}{\left(x^2 + z^2\right)^{\frac{s}{2}}\left(\mathrm{e}^{2\pi x} - 1\right)}\,\mathrm{d}x,$$

where $s \in \mathbb{C} \setminus \{1\}$.

2. (6 points) Consider a sphere with centre at the origin $(0, 0, 0) \in \mathbb{R}^3$ and a radius of $\sqrt{2}$, as shown below.



Utilising the transformation from Cartesian coordinates to spherical coordinates, given by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, and $\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\mathrm{d}\theta\,\mathrm{d}\phi$, for $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, calculate the volume of the sphere by means of an integral.

3. (6 points) Given that $\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$. Evaluate the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-(ax^2+bx+c)}\,\mathrm{d}x,$$

where $\Re a > 0$.

4. (6 points) Evaluate the integral

$$\int_0^\infty \frac{\sin x \cos x \tan x}{x^2}\,\mathrm{d}x.$$

5. Evaluate the integrals

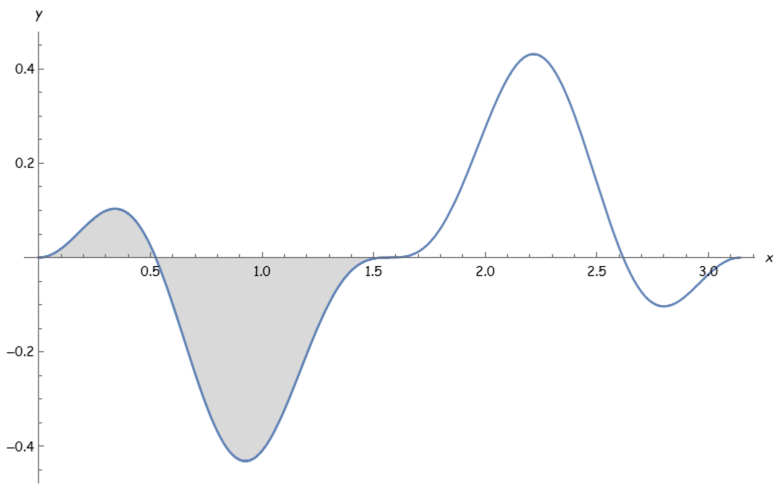
(a) (3 points)

$$\int_0^1 \frac{1}{t^4+1}\,\mathrm{d}t.$$

(b) (3 points)

$$\int_0^1 \frac{t^2-1}{t^4+4t^2+1}\,\mathrm{d}t.$$

6. (6 points) The graph of $\cos x \sin(2x) \cos(3x) \sin x$ from $0 \le x \le \pi$ is shown below.



The shaded region represents the area from $0 \le x \le \frac{\pi}{2}$. Determine the area of this shaded region.

7. (6 points) Establish the equality

$$\lim_{n\to\infty}\left(\frac{\pi^2}{n^2}\sin^2\left(\frac{\pi}{n}\right) + \frac{2\pi^2}{n^2}\sin^2\left(\frac{2\pi}{n}\right) + \cdots + \frac{(n-1)\pi^2}{n^2}\sin^2\left(\pi - \frac{\pi}{n}\right)\right) = \frac{\pi^2}{4}.$$

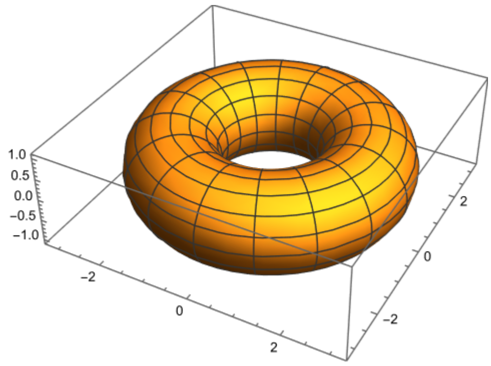
8. (6 points) Let $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, where $\alpha \neq \beta$. If we define

$$\Lambda(\alpha,\beta,\lambda) = \int_0^1 (\beta x + \alpha(1-x))^\lambda\,\mathrm{d}x,$$

show that

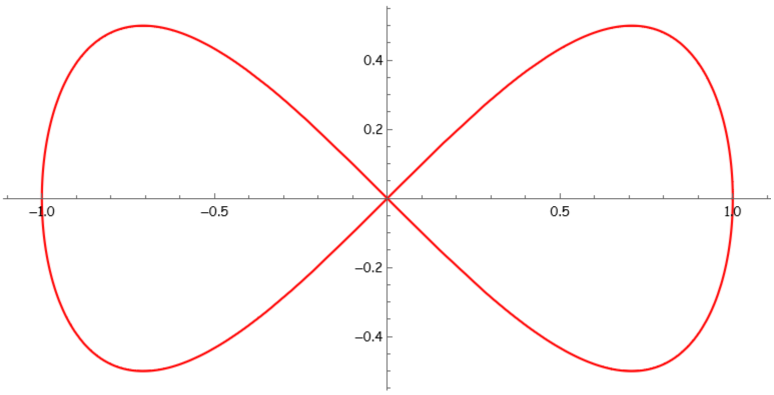
$$\lim_{\lambda\to 0}(\Lambda(\alpha,\beta,\lambda))^{\frac{1}{\lambda}} = \lim_{\lambda\to 0}(\Lambda(\beta,\alpha,\lambda))^{\frac{1}{\lambda}} = \frac{\beta^{\frac{\beta}{\beta-\alpha}}}{\mathrm{e}\,\alpha^{\frac{\alpha}{\beta-\alpha}}}.$$

9. (6 points) The torus shown below is generated by rotating the circular region defined by $(x-2)^2 + z^2 = 1$ about the z-axis.



Determine the volume of the torus by means of an integral.

10. (6 points) In algebraic geometry, the lemniscate of Gerono is a plane algebraic curve of degree four, resembling an ∞ symbol or figure eight. The parametric equations for this curve are $x(\varphi) = \cos\varphi$ and $y(\varphi) = \sin\varphi\cos\varphi$. The plot of the curve is shown below.



Establish that the length of the curve for φ in the interval $[-1, 1]$ is given by

$$\int_{-1}^{1} \sqrt{1 - \frac{1}{2}\cos(2\varphi) + \frac{1}{2}\cos(4\varphi)}\, \mathrm{d}\varphi \approx 1.63077.$$

Note: The integral itself should not be evaluated.