UNIVERSITY OF IBADAN

ODE

INTEGRATION BEE (2.0)

UI ODE-Integration Bee
Solutions to the Qualifying Stage (100 L)

*"I remember once going to see him [Ramanujan] when he was lying ill at Putney.
I had ridden in taxi-cab No. 1729, and remarked that the number seemed to me rather a dull one, and that i hoped it was not an unfavorable omen.
No, he replied,* it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways*"*. G. H. Hardy

1. (8 points) The Riemann zeta function, $\zeta(s)$, defined for $\Re(s) > 1$ by
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$
is an extremely important special function in mathematics an physics, arising in definite integration and intimately connected with very deep results surrounding the prime number theorem. Notably, the domain of $\zeta(s)$ can be extended to the entire complex plane $\mathbb{C}$, with the exception of $s = 1$, via the Hermite integral representation, which is a direct application of the Abel-Plana formula. For $\Re(s) > 1$, an integral representation relating $\zeta(s)$ to the gamma function is
$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} \, \mathrm{d}x,$$
where $\Gamma(s)$ denotes the gamma function, defined as $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, \mathrm{d}t$. Some values of $\zeta(s)$ for even s are
$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450}.$$
A generalization of $\zeta(s)$ for even s is given in terms of Bernoulli numbers.
In Abdulsalam's article titled "*New Closed Forms for a Dilogarithmic Integral, Related Integrals, and Series,*" he proved that
$$\sum_{k=1}^{\infty} \left(\frac{1}{k} \int_0^{\infty} \frac{x}{(x^2 + k^2)^2 (e^{2\pi x} - 1)} \, \mathrm{d}x \right) = \frac{\pi^4}{288} - \frac{\zeta(3)}{4}.$$
Justify the convergence of this series.

Solution 0.1. *Clearly,*
$$(x^2 + k^2)^2 > (0^2 + k^2)^2 = k^4 \quad \forall \, x \in (0, \infty).$$
Utilizing the integral relating $\zeta(s)$ and $\Gamma(s)$, we have
$$\frac{1}{k} \int_0^{\infty} \frac{x}{(x^2 + k^2)^2 (e^{2\pi x} - 1)} \, \mathrm{d}x < \frac{1}{k^5} \int_0^{\infty} \frac{x}{e^{2\pi x} - 1} \, \mathrm{d}x = \frac{1}{k^5} \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{x}{e^x - 1} \, \mathrm{d}x$$
$$= \frac{1}{k^5} \frac{1}{4\pi^2} \zeta(2) \Gamma(2) = \frac{1}{24 k^5}, \quad \forall \, k \geq 1.$$
Since $\sum_{k=1}^{\infty} \frac{1}{24k^5} = \frac{\zeta(5)}{24}$, by comparison test
$$\sum_{k=1}^{\infty} \left(\frac{1}{k} \int_0^{\infty} \frac{x}{(x^2 + k^2)^2 (e^{2\pi x} - 1)} \, \mathrm{d}x \right)$$
converges.

2. (6 points) Define a sequence $\{\zeta_n\}_{n=0}^{\infty}$ by $\zeta_0 = 1$, $\zeta_1 = 8$ and $\zeta_n = 2\zeta_{n-1} + \zeta_{n-2}$ for $n \geq 2$. The infinite sum
$$\sum_{n=1}^{\infty} \int_0^{\frac{2021\pi}{14}} \sin(\zeta_{n-1} x) \sin(\zeta_n x) \, \mathrm{d}x$$
can be expressed as an irreducible fraction $\frac{p}{q}$. Compute $p + q$.

Solution 0.2. *Observe the following Product-to-Sum Identity:*
$$\sin(\zeta_{n-1} x) \sin(\zeta_n x) = \frac{1}{2} [\cos((\zeta_n - \zeta_{n-1}) x) - \cos((\zeta_n + \zeta_{n-1}) x)]$$

So that,

$$\begin{aligned}\int_0^{k\pi} \sin(\zeta_{n-1}x)\sin(\zeta_n x)\,\mathrm{d}x &= \frac{1}{2}\int_0^{k\pi} \cos((\zeta_n-\zeta_{n-1})x)-\cos((\zeta_n+\zeta_{n-1})x)\,\mathrm{d}x\\ &= \frac{1}{2}\left(\frac{\sin(k(\zeta_n-\zeta_{n-1})\pi)}{\zeta_n-\zeta_{n-1}}-\frac{\sin(k(\zeta_n+\zeta_{n-1})\pi)}{\zeta_n+\zeta_{n-1}}\right)\end{aligned}$$

Where $k=\frac{2021}{14}$. *From the recurrence relation* $\zeta_n = 2\zeta_{n-1}+\zeta_{n-2}$, *we have that* $\zeta_n-\zeta_{n-1}=\zeta_{n-1}+\zeta_{n-2}$, *so for all* $n\geq 2$, *we have the telescoping sum*

$$\begin{aligned}\sum_{n=2}^{\infty}\int_0^{k\pi} \sin(\zeta_{n-1}x)\sin(\zeta_n x)\,\mathrm{d}x &= \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{\sin(k(\zeta_n+\zeta_{n-2})\pi)}{\zeta_n+\zeta_{n-2}}-\frac{\sin(k(\zeta_n+\zeta_{n-1})\pi)}{\zeta_n+\zeta_{n-1}}\right)\\ &= \frac{1}{2}\left(\frac{\sin(k(\zeta_1+\zeta_0)\pi)}{\zeta_1+\zeta_0}-\lim_{n\to\infty}\frac{\sin(k(\zeta_n+\zeta_{n-1})\pi)}{\zeta_n+\zeta_{n-1}}\right)=\frac{1}{2}\frac{\sin(k(\zeta_1+\zeta_0)\pi)}{\zeta_1+\zeta_0}\end{aligned}$$

Then,

$$\begin{aligned}&\sum_{n=1}^{\infty}\int_0^{k\pi} \sin(\zeta_{n-1}x)\sin(\zeta_n x)\,\mathrm{d}x\\ &= \frac{1}{2}\left(\frac{\sin(k(\zeta_1-\zeta_0)\pi)}{\zeta_1-\zeta_0}-\frac{\sin(k(\zeta_1+\zeta_0)\pi)}{\zeta_1+\zeta_0}\right)+\frac{1}{2}\frac{\sin(k(\zeta_1+\zeta_0)\pi)}{\zeta_1+\zeta_0}=\frac{1}{2}\frac{\sin(k(\zeta_1-\zeta_0)\pi)}{\zeta_1-\zeta_0}\end{aligned}$$

We now plug in the values of ζ_0 *and* ζ_1 *with* k, *then we conclude that*

$$\frac{1}{2}\frac{\sin(k(\zeta_1-\zeta_0)\pi)}{\zeta_1-\zeta_0}=\frac{1}{14}$$

Therefore $p=1$ *and* $q=14$ *and so we have* $p+q=15$.

3. (6 points) Evaluate

$$\int_0^1 \frac{x}{x\sqrt{x}+1}\,\mathrm{d}x$$

Solution 0.3. *Set* $u=\sqrt{x}$ *and by partial fractions, we get*

$$\begin{aligned}\int_0^1 \frac{x}{x\sqrt{x}+1}\,\mathrm{d}x = \int_0^1 \frac{2u^3}{u^3+1}\,\mathrm{d}u &= 2\int_0^1\left(1-\frac{1/3}{u+1}-\frac{(-1/3)u+2/3}{u^2-u+1}\right)\mathrm{d}u\\ &= 2\left(1-\frac{\log 2}{3}\right)+\frac{2}{3}\int_0^1\frac{u-2}{u^2-u+1}\,\mathrm{d}u\end{aligned}$$

For the later integral, write $v=u^2-u+1$ *and thus,*

$$\int\frac{u-2}{u^2-u+1}\,\mathrm{d}u=\frac{1}{2}\log(u^2-u+1)-\frac{3}{2}\int\frac{\mathrm{d}u}{(u-1/2)^2+3/4}$$

The remaining integral can be resolved using the arctan *identity, by so doing we have that*

$$\int_0^1\frac{x}{x\sqrt{x}+1}\,\mathrm{d}x = 2-\frac{2}{3}\log 2-\frac{2\pi}{9}\sqrt{3}$$

4. (6 points) Compute the value of $20a+b$, where a and b are positive integers satisfying

$$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x=\frac{\pi^b}{a}$$

Solution 0.4. *Let* $u = \cos x$*, so that*

$$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,\mathrm{d}x = \int_0^1 \frac{\arccos(-u)+\arccos(u)}{1+u^2}\,\mathrm{d}u$$

Using the fact that $\arccos(u)+\arccos(-u)=\pi$*, we observe that*

$$\int_0^1 \frac{\arccos(-u)+\arccos(u)}{1+u^2}\,\mathrm{d}u = \pi\int_0^1 \frac{\mathrm{d}u}{1+u^2} = \frac{\pi^2}{4}$$

So we have $a=4$ *and* $b=2$*, which implies that* $20a+b=82$*.*

5. (6 points) Given

$$\int_0^8 \frac{1}{\sqrt{1+\sqrt{1+x}}}\,\mathrm{d}x = \frac{a+b\sqrt{2}}{c}$$

such that $\gcd(a,b,c)=1$. Find the value of $a+b+c$. (Note: $\gcd(a,b)$ means the greatest common divisor of a and b).

Solution 0.5. *Let* $u = x+1$ *and thus,*

$$\int_0^8 \frac{1}{\sqrt{1+\sqrt{1+x}}}\,\mathrm{d}x = \int_1^9 \frac{1}{\sqrt{\sqrt{u}+1}}\,\mathrm{d}u = 2\int_2^4 \left(\sqrt{u}-\frac{1}{\sqrt{u}}\right)\mathrm{d}u$$

Finally,

$$\int_0^8 \frac{1}{\sqrt{1+\sqrt{1+x}}}\,\mathrm{d}x = \frac{8+4\sqrt{2}}{3}$$

Hence $a+b+c=15$*.*

6. (6 points) Compute

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\pi e}}\,\mathrm{d}x$$

Hint: Use the substitution $x=\frac{\pi}{2}-y$.

Solution 0.6.

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\pi e}}\,\mathrm{d}x = -\int_{\frac{\pi}{2}}^0 \frac{\mathrm{d}x}{1+\tan\left(\frac{\pi}{2}-x\right)} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+(\cot x)^{\pi e}} = \int_0^{\frac{\pi}{2}} \frac{(\tan x)^{\pi e}}{1+(\tan x)^{\pi e}}\,\mathrm{d}x$$

It's clear that,

$$\begin{aligned} 2\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\pi e}}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \left(\frac{1}{1+(\tan x)^{\pi e}}+\frac{(\tan x)^{\pi e}}{1+(\tan x)^{\pi e}}\right)\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{1+(\tan x)^{\pi e}}{1+(\tan x)^{\pi e}}\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2} \end{aligned}$$

So that,

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\pi e}}\,\mathrm{d}x = \frac{\pi}{4}$$

7. (6 points) If f is continuously differentiable on $[0, a]$ for $a > 0$, and $f(a) = f(0) = b$, prove that

$$\int_0^a (f'(x))^2 \, \mathrm{d}x \geq 2 \int_0^a f(x) \, \mathrm{d}x - \left(2ab + \frac{a^3}{12}\right)$$

Hint: Consider the inequality $0 \leq \int_0^a \left(f'(x) + x - \frac{a}{2}\right)^2 \, \mathrm{d}x$.

Solution 0.7. *Observe that,*

$$0 \leq \int_0^a \left(f'(x) + x - \frac{a}{2}\right)^2 \, \mathrm{d}x$$
$$0 \leq \int_0^a \left(\left(f'(x)\right)^2 + 2xf'(x) - af'(x) - ax + x^2 + \frac{a^2}{4}\right) \mathrm{d}x$$

And so,

$$\begin{aligned}
-\int_0^a \left(f'(x)\right)^2 \, \mathrm{d}x &\leq \int_0^a 2xf'(x) \, \mathrm{d}x + \int_0^a \left(-af'(x) - ax + x^2 + \frac{a^2}{4}\right) \mathrm{d}x \\
&\leq 2\left(ab - \int_0^a f(x) \, \mathrm{d}x\right) + \left(-ab - \frac{a^3}{2} + \frac{a^3}{3} + \frac{a^3}{4}\right) + ab \\
&= -2\int_0^a f(x) \, \mathrm{d}x + 2ab + \frac{a^3}{12}
\end{aligned}$$

We conclude,

$$\int_0^a (f'(x))^2 \, \mathrm{d}x \geq 2 \int_0^a f(x) \, \mathrm{d}x - \left(2ab + \frac{a^3}{12}\right)$$

8. (4 points) Determine whether the integral converges or diverges;

$$\int_{-\infty}^{\infty} \frac{1}{4 + x^2} \, \mathrm{d}x$$

Solution 0.8.

$$\int_{-\infty}^{\infty} \frac{1}{4 + x^2} \, \mathrm{d}x = 2 \lim_{N \to \infty} \int_0^N \frac{1}{4 + x^2} \, \mathrm{d}x = \lim_{N \to \infty} \arctan\left(\frac{x}{2}\right)\Big|_0^N = \frac{\pi}{2}$$

Thus, the integral converges.

9. (4 points) Compute

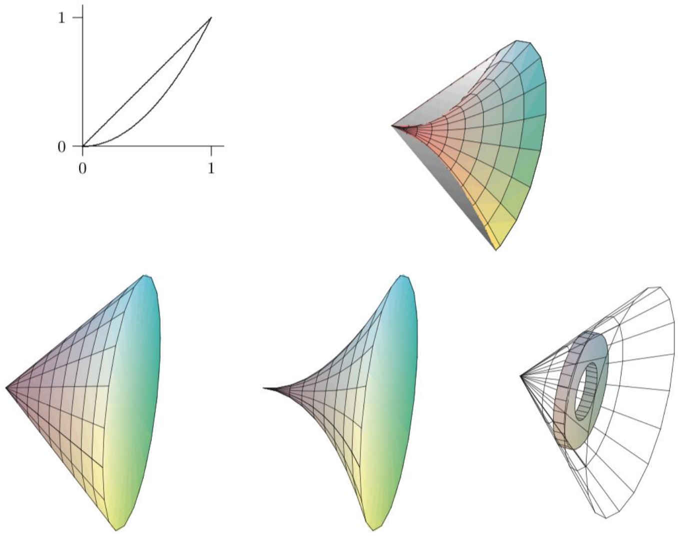
$$\int_1^{\infty} \arctan\left(\frac{1}{x}\right) \mathrm{d}x$$

Solution 0.9.

$$\begin{aligned}
\int_1^{\infty} \arctan\left(\frac{1}{x}\right) \mathrm{d}x &= \lim_{N \to \infty} \int_1^N \arctan\left(\frac{1}{x}\right) \mathrm{d}x = \lim_{N \to \infty} \left(x \arctan(1/x) + \log\sqrt{x^2 + 1}\right)\Big|_1^N \\
&= \lim_{N \to \infty} \left(n \arctan(n/x) + \log\sqrt{n^2 + 1} - \frac{\pi}{4} - \log\sqrt{2}\right) \\
&= \infty
\end{aligned}$$

We conclude the integral diverges.

10. (8 points) Find the volume of the object generated when the area between $y = x^2$ and $y = x$ is rotated around the x-axis. This solid has a hole in the middle; we can compute the volume by subtracting the volume of the hole from the volume enclosed by the outer surface of the solid.
In the figure below we show the region that is rotated, the resulting solid with the front half cut away, the cone that forms the outer surface, the horn-shaped hole, and a cross-section perpendicular to the x-axis.



Solution 0.10. *Notice, the point of intersection is at $x = 0$ and $x = 1$. Now the volume of the solid generated is found by applying the Washer method. That is,*

$$\textit{Volume} = \int_0^1 \pi(x^2 - x^4)\,\mathrm{d}x = \pi \int_0^1 (x^2 - x^4)\,\mathrm{d}x = \frac{2\pi}{15} \text{ \textit{cubic units.}}$$

"An equation means nothing to me,
unless it expresses a thought of God". Srinivasa Ramanujan.