UNIVERSITY OF IBADAN

ODE

INTEGRATION BEE (2.0)

UI ODE-Integration Bee
Final Stage

Instructions for Participants

Congratulations on qualifying for the UI ODE-Integration Bee Final Stage!. Please carefully read and follow these instructions:

1. Answer all questions.
2. Write your responses legibly and concisely. Use a clear and neat handwriting.
3. Use only the provided sheets for your answers. Ensure that your solutions are well-structured and organized.
4. Write your full name and matriculation number at the top of each page of your answer sheet.
5. Follow any specific instructions provided with individual questions.
6. Do not waste too much time on a question.
7. Be mindful of time. You will have 2 hours 30 minutes for the entire test.
8. If you have any questions or require clarification during the test, please raise your hand and wait for an invigilator to assist you.
9. Electronic devices, calculators, books, and any unauthorized aids are strictly prohibited during the test.
10. Maintain academic integrity. Do not discuss the content of the test with your fellow participants until the test is over.

This Final Stage aims to evaluate your understanding and problem-solving skills in integration and it’s application. Good luck!.

Questions

Information for participants: The maximum points attainable for this test is 60 points. Take your time to read each questions carefully before you provide answers to them.

1. (8 points) The Euler-Mascheroni constant γ defined by

$$\gamma := \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n$$

   is regarded as an important constant of analysis, shadowed only by e and π in significance. Havil (2003) provides an excellent history for this constant.

   Now you will follow De Temple (1993) to prove that a small modification of the sequence defining the Euler's constant, produces a sequence that converges to γ with error of the order $1/n^2$.
   Let

$$b_n := \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2}\right).$$

   Then

$$\frac{1}{24(n+1)^2} < b_n - \gamma < \frac{1}{24n^2}$$

   so that

$$\lim_{n\to\infty} n^2(b_n - \gamma) = \frac{1}{24}.$$

2. (6 points) Suppose

$$N_{0,4}(a;m) = \int_0^\infty \frac{\mathrm{d}x}{(x^4 + 2ax^2 + 1)^{m+1}}$$

$$N_{0,4}(q_4, q_2, q_0; m) = \int_0^\infty \frac{x^2\mathrm{d}x}{(q_4x^4 + q_2x^2 + q_0)^{m+1}}.$$

   Show the following identities holds true;

   a.

$$N_{0,4}(q_4, q_2, q_0; m) := \frac{1}{q_0^{m+3/4} q_4^{1/4}} N_{0,4}\left(\frac{q_2}{2\sqrt{q_0q_4}}; m\right)$$

   b.

$$N_{0,4}(1; m) = \frac{\pi}{2^{4m+3}} \binom{4m+2}{2m+1}.$$

3. (6 points) Let

$$f(n) := \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \sqrt{2n+1}$$

   and check that

$$f(n) < \int_0^\infty e^{u^2/2}\,\mathrm{d}u < \left(1 + \frac{1}{2n+1}\right) f(n)$$

   Conclude that

$$\int_0^\infty e^{-u^2/2}\,\mathrm{d}u = \lim_{n\to\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \sqrt{2n+1}.$$

4. (6 points) Prove for $q > 0$ and $a \in \mathbf{R}$;

$$\int_{-a}^{\infty} \frac{e^{-qx}}{\sqrt{x+a}} = \sqrt{\frac{\pi}{q}} e^{aq}.$$

5. (6 points) Let $f(x) > 0$ be a continuous and decreasing function on $[a, b]$. Then the inequality

$$\frac{\displaystyle\int_a^b f^{\beta}(x)\,\mathrm{d}x}{\displaystyle\int_a^b f^{\gamma}(x)\,\mathrm{d}x} \geq \frac{\displaystyle\int_a^b (x-a)^{\alpha} f^{\beta}(x)\,\mathrm{d}x}{\displaystyle\int_a^b (x-a)^{\alpha} f^{\gamma}(x)\,\mathrm{d}x}$$

holds for every positive real number $\alpha > 0$ and $\beta \geq \gamma > 0$.

6. (6 points) Find the exact form of the integral

$$\int_0^1 \frac{x^2 \cosh(\arctan x)}{\sqrt{1+x^2}}\,\mathrm{d}x.$$

7. (6 points) Evaluate

$$\int_0^{\infty} \cos\left(\frac{x}{\pi} - \frac{e}{x}\right)^2 \mathrm{d}x.$$

8. (4 points) Prove or Disprove the assertion;

$$\lim_{n\to\infty} \frac{1}{n^2}\left(\sin^3\frac{\pi}{4n} + 2\sin^3\frac{2\pi}{4n} + \cdots + n\sin^3\frac{n\pi}{4n}\right) = \frac{\sqrt{2}}{9\pi^2}(52 - 15\pi).$$

9. (4 points) Show for $a, b > 0$,

$$\int_0^{\infty} \frac{\log(1+a^2x^2)}{1+b^2x^2}\,\mathrm{d}x = \frac{\pi}{b}\log\left(1+\frac{a}{b}\right).$$

10. (8 points)

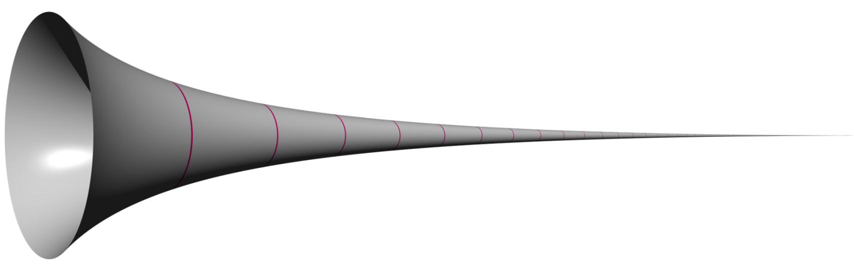

Figure 1: Gabriel's Horn

Gabriel's Horn is defined to be the region obtained by rotating the region below the graph of $f(x) = 1/x$ from $x \geq 1$ about the $x-$axis.

Show that the horn has finite volume and infinite surface area. In other words, "Angel Gabriel's horn can be filled with paint, but it cannot be painted!".